Since f is arbitrarily taken with $f \prec G$, we obtain

$$\mu_0(G) \leq \sum_{j=1}^{\infty} \mu_0(G_j)$$

Hence

$$\mu(E) \leq \mu_0(G) \leq \sum_{j=1}^{\infty} \mu_0(G_j)$$

$$\leq \sum_{j=1}^{\infty} \left(\mu(E_j) + \frac{\varepsilon}{2^j} \right)$$

$$= \left(\sum_{j=1}^{\infty} \mu(E_j) \right) + \varepsilon$$

Letting $\varepsilon \to 0$ gives

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

Step 2. μ is a Borel measure.

Equivalently, we need to show that all open sets are μ-measurable.

Let $U \subset X$ be open. We need to prove

$$\mu(C) \geq \mu(C \cap U) + \mu(C \setminus U), \ \forall C \subset X.$$

By the definition of μ, it is enough to prove

$$(*) \quad \mu(G) \geq \mu(G \cap U) + \mu(G \setminus U), \ \forall \text{ open } G$$

(because if this is true, then $\forall \varepsilon > 0$, pick open $G \supset C$ such that $\mu(C) \geq \mu(G) - \varepsilon$. Then by $(*)$,

$$\mu(C) \geq \mu(G) - \varepsilon \geq \mu(G \cap U) + \mu(G \setminus U) - \varepsilon$$
$$\geq \mu(C \cap U) + \mu(C \setminus U) - \varepsilon.)$$

To prove (*), we may assume $\mu(G) < \infty$.
Let $\varepsilon > 0$, and pick $\varphi < G \cap U$ such that

$$\Lambda(\varphi) \geq \mu_0(G \cap U) - \varepsilon.$$

Let $K = \mathrm{supp}(\varphi)$. Pick $\psi < G \setminus K$.

Since $\mathrm{supp}(\psi)$ and K are disjoint,

$$\varphi + \psi < G.$$

Hence

$$\begin{aligned} \mu(G) = \mu_0(G) &\geq \Lambda(\varphi + \psi) \\ &= \Lambda(\varphi) + \Lambda(\psi) \\ &\geq \mu(G \cap U) - \varepsilon + \Lambda(\psi). \end{aligned}$$

Recall that $\psi < G \setminus K$ is arbitrily taken,
we obtain

$$\mu(G) \geq \mu(G \cap U) - \varepsilon + \mu(G \setminus K)$$

$$\geq \mu(G \cap U) - \varepsilon + \mu(G \setminus U)$$

(since $G \setminus K \supset G \setminus U$).

Letting $\varepsilon \to 0$ gives

$$\mu(G) \geq \mu(G \cap U) + \mu(G \setminus U).$$

Step 3. μ is finite on compact sets.

We shall prove

$$\mu(K) = \inf\{\Lambda(f) : K \prec f\}$$

for all compact sets K,

where $K \prec f$ means $f \in C_c(X)$,

$0 \leq f \leq 1$ on X and $f = 1$ on K.

We first show $\mu(K) \leq \inf\{\Lambda(f) : K \prec f\}$.

Let $f \in C_c(X)$ such that $K \prec f$.

For $\alpha \in (0,1)$, define

$$G_\alpha = \{x \in X : f(x) > \alpha\}.$$

Then G_α is open and $G_\alpha \supset K$.

Let $\varphi \prec G_\alpha$. Then

$$\varphi < \frac{f}{\alpha} \quad \text{on } G_\alpha$$

Hence

$$\varphi \leq \frac{f}{\alpha} \quad \text{on } X.$$

It follows that

$$\Lambda(\varphi) \leq \Lambda\left(\frac{f}{\alpha}\right) = \frac{1}{\alpha}\Lambda(f)$$

(here we used the positivity of Λ)

Hence

$$\mu(G_\alpha) \leq \frac{1}{\alpha}\Lambda(f).$$

In particular

$$\mu(K) \leqslant \mu(G_\alpha) \leqslant \frac{1}{\alpha} \Lambda(f).$$

Letting $\alpha \uparrow 1$ gives $\mu(K) \leqslant \Lambda(f)$.

This showes $\mu(K) \leqslant \inf\{\Lambda(f) : K < f\}$.

To show the other direction, $\forall \varepsilon > 0$, we can find open $G \supset K$ such that

$$\mu(K) \geqslant \mu_0(G) - \varepsilon = \mu(G) - \varepsilon.$$

By Urysohn's lemma, $\exists\ f \in C_c(X)$ such that

$$K < f < G.$$

Hence

$$\begin{aligned} \mu(K) &\geqslant \mu(G) - \varepsilon \\ &\geqslant \Lambda(f) - \varepsilon. \\ &\geqslant \inf\{\Lambda(g) : K < g\} - \varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$ gives the desired inequality,

Step 4. $\Lambda(f) = \int f \, d\mu, \ \forall f \in C_c(X)$.

Actually we only need to prove

(**) $\Lambda(f) \leq \int f \, d\mu, \ \forall f \in C_c(X)$.

(because the other direction follows by replacing f by $-f$ in the above inequality)

Let $f \in C_c(X)$. Then $f(X) \subset [a,b]$ for some $a, b \in \mathbb{R}$.

Let $\varepsilon > 0$. pick

$$y_0 < a < y_1 < y_2 < \cdots < y_n = b$$

such that $y_{i+1} - y_i < \varepsilon$.

Let $K = \text{supp}(f)$. Let

$$E_j = f^{-1}(y_{j-1}, y_j] \cap K, \quad j=1,\dots,n$$

Then E_j are disjoint, measurable,

and $$\bigcup_{j=1}^{n} E_j = K.$$

Since K is compact, $\mu(K) < \infty$ so are $\mu(E_j)$.

Next we pick open $G_j \supset E_j$ such that

① $G_j \subset \{x : y_{j-1} - \varepsilon < f(x) < y_j + \varepsilon\}$

② $\mu(E_j) \geq \mu(G_j) - \frac{\varepsilon}{n}$.

Notice $K = \bigcup_{j=1}^{n} E_j \subset \bigcup_{j=1}^{n} G_j$.

Hence $\exists$ $\varphi_j \prec G_j$ with $\sum_{j=1}^{n} \varphi_j = 1$ on K.

Hence

$$f = \sum_j f \cdot \varphi_j \quad \text{on } X$$

Hence

$$\Lambda(f) = \sum_{j=1}^{n} \Lambda(f\varphi_j)$$

$$\leq \sum_{j=1}^{n} \Lambda\left((y_j+\varepsilon)\varphi_j\right)$$

(because $f < y_j + \varepsilon$ on G_j)

$$= \sum_{j=1}^{n} (y_j+\varepsilon)\,\Lambda(\varphi_j).$$

$$= \sum_{j=1}^{n} \left(|a| + y_j + \varepsilon\right) \Lambda(\varphi_j) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)$$

$$< \sum_{j=1}^{n} \left(|a| + y_j + \varepsilon\right) \mu(G_j) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)$$

(since $|a| + y_j + \varepsilon > 0$)

$$\leq \sum_{j=1}^{n} \left(|a| + y_{j-1} + 2\varepsilon\right) \cdot \left(\mu(E_j) + \frac{\varepsilon}{n}\right) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)$$

$$\leq \sum_{j=1}^{n} y_{j-1}\mu(E_j) + |a|\cdot\left(\sum_{j=1}^{n}\mu(E_j) - \sum_{j=1}^{n}\Lambda(\varphi_j)\right) + O(\varepsilon).$$

$$\leq \int f\,d\mu + O(\varepsilon).$$

(since $\sum_j y_{j-1}\chi_{E_j} \leq \int f\,d\mu$)

Here we used the facts that $\sum_{j=1}^{n}\mu(E_j) = \mu(K)$

and $\mu(K) \leq \Lambda\left(\sum_{j=1}^{n}\varphi_j\right)$

(since $K < \sum_{j=1}^{n}\varphi_j$, we use Step 3).